BIRATIONAL GEOMETRY OF QUARTIC 3-FOLDS II: THE IMPORTANCE OF BEING $\mathbb{Q}$-FACTORIAL

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Abstract. The paper explores the birational geometry of terminal quartic 3-folds. In doing this I develop a new approach to study maximal singularities with positive dimensional centers. This allows to determine the pliability of a $\mathbb{Q}$-factorial quartic with ordinary double points, and it shows the importance of $\mathbb{Q}$-factoriality in the context of birational geometry of uniruled 3-folds.

Introduction

Let X be a uniruled 3-fold, then X is generically covered by rational curves. It is a common belief that both biregular and birational geometry of X are somehow governed by these families of rational curves. In this paper I am interested in birational geometry of these objects. The Minimal Model Program states that such a X is birational to a Mori fiber Space (Mfs). Roughly saying after some birational modification either X can be fibered in rational surfaces or rational curves or it becomes Fano. For a comprehensive introduction to this realm of ideas as well as for the basic definitions and results see [CR] and [Co2].

In the attempt to tidy up the birational geometry of 3-fold Mori fiber Spaces we introduced the notion of pliability, [CM].

Definition 1 (Corti)**.** If X is an algebraic variety, we define the *pliability* of X to be the set

$$\mathcal{P}(X) = \big\{\text{Mfs } Y \to T \mid Y \text{ is birational to } X\big\}/\text{square equivalence.}$$

We say that X is *birationally rigid* if $\mathcal{P}(X)$ consists of one element.

It is usually quite hard to determine the pliability of a given Mori Space, and not many examples are known. The first rigorous result dates back to Iskovskikh and Manin, [IM]. The main theorem of [IM] states, in modern terminology, that any birational map $\chi : X \dashrightarrow Y$ from a smooth quartic $X \subset \mathbb{P}^4$ to a Mori fiber space is an isomorphism. This means that $\mathcal{P}(X) = \{X\}$ and X is birationally rigid.

On the other hand consider a quartic threefold $X \subset \mathbb{P}^4$ defined by $\det M = 0$, where M is a 4×4 matrix of linear forms. One can define a map $f : X \dashrightarrow \mathbb{P}^3$ by the assignment $P \mapsto (x_0 : x_1 : x_2 : x_3)$, where (x_0, x_1, x_2, x_3) is a solution of the system of linear equations obtained substituting the coordinates of P in M. For M sufficiently general such a map is well defined and birational. In this case f gives a rational parameterization of X. The singularities of X correspond to points where the rank drops. It is not difficult to show that, for a general M, the corresponding

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quartic has only ordinary double points corresponding to points where the rank is 2. Thus a general determinantal quartic threefold has only ordinary double points and it is rational.

From the pliability point of view this is discouraging. Minimal Model Theory requires to look at terminal $\mathbb{Q}$-factorial 3-folds and ordinary double points are the simplest possible terminal singularities. It would be unpleasant if a bunch of ordinary double points were to change a rigid structure to a rational variety. The point I want to stress in this paper is that the rationality of a determinantal quartic is due to the lack of $\mathbb{Q}$-factoriality and not to the presence of singularities.

Theorem 2. *Let $X_4 \subset \mathbb{P}^4_{\mathbb{C}}$ be a $\mathbb{Q}$-factorial quartic 3-fold with only ordinary double points as singularities. Then X is neither birationally equivalent to a conic bundle nor to a fibration in rational surfaces. Every birational map $\chi : X \dashrightarrow Y$ to a Fano 3-fold is a self map, that is $Y \cong X$, in particular X is not rational. This is to say that X is birationally rigid.*

Remark 3. *The case of a general quartic with one ordinary double point has been treated by Pukhlikov,* [Pu]*. Observe that in this case X is automatically $\mathbb{Q}$-factorial. More recently Grinenko studied the case of a general quartic containing a plane.*

A variety is said to be $\mathbb{Q}$-factorial if every Weil divisor is $\mathbb{Q}$-Cartier. Such an innocent definition is quite subtle when realized on a projective variety. It does depend both on the kind of singularities of X and on their position. To my knowledge there are very few papers that tried to shed some light on this question, [Cl] [We]. In the case of a Fano 3-fold, $\mathbb{Q}$-factorial is equivalent to $\dim H^2(X,\mathbb{Z}) = \dim H_4(X,\mathbb{Z})$, a global topological property, invariant for diffeomorphic Fano 3-folds. A recent paper of Ciliberto and Di Gennaro, [CDG], deals with hypersurfaces with few nodes. The general behavior is that the presence of few nodes does not break $\mathbb{Q}$-factoriality. This is not true even for slightly worse singularities, as the following example shows.

Example 4 (Kollár)**.** *Consider the linear system Σ, of quartics spanned by the following set of monomials $\{x_0^4, x_1^4, (x_4^2x_3 + x_2^3)x_0, x_3^3x_1, x_4^2x_1^2\}$. Then a general quartic $X \in \Sigma$ has a unique singularity P at $(0:0:0:0:1)$ and the quadratic term is a general quadric in the linear system spanned by $\{x_3x_0, x_1^2\}$, so that analytically $P \in X \sim (0 \in (xy + z^2 + t^l = 0))$ and P is a cA_1 point. The 3-fold X is not $\mathbb{Q}$-factorial since the plane $\Pi = (x_0 = x_1 = 0)$ is contained in X. The idea is that a general quartic containing a plane has 9 ordinary double points, the intersection of the two residual cubics. In the above case the two cubics intersect just in the point P.*

There is a slightly stronger version of Theorem 2.

Theorem 5. *Let $X_4 \subset \mathbb{P}^4_k$ be a $\mathbb{Q}$-factorial quartic 3-fold with only ordinary double points as singularities over a field k, not necessarily algebraically closed, of characteristic 0. Then $\mathcal{P}(X) = \{X\}$.*

If one considers non algebraically closed fields then peculiar aspects of factoriality and its relation with birational rigidity appear. Theorem 5 and its significance in this contest, were suggested by János Kollár.

Example 6. *Consider the following quartic Z*

$$(x_0^2 + x_1^2)^2 + (x_2^2 + x_3^2)^2 + x_4C = 0.$$

Then Z is not $\mathbb{Q}$-factorial over $\mathbb{C}$ because $(x_4 = 0)_{|Z}$ is a pair of quadrics, say Q and $\overline{Q}$. For a general cubic C the singular points of Z are twelve distinct ordinary double points. The existence of Minimal Model Program for 3-folds implies that Z is birational to some Mori Space $Y \neq Z$. Indeed this is the midpoint of a Sarkisov link. This can be easily seen with the unprojection method developed by Reid, [Un]. *The equation of Z is*

$$Q\overline{Q} + HC = 0.$$

We can introduce the two ratios $y = Q/H = -C/\overline{Q}$ and $z = \overline{Q}/H = -C/Q$. These are both of degree one and unproject Z to the following complete intersections

$$X = \left\{ \begin{array}{rl} yH = & Q \\ y\overline{Q} = & -C \end{array} \right. \subset \mathbb{P}^5 \quad X' = \left\{ \begin{array}{rl} zH = & \overline{Q} \\ zQ = & -C \end{array} \right. \subset \mathbb{P}^5$$

X and X' are projectively equivalent, thus we have a Sarkisov self link, see [Co2],

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] \arrow[rr, dotted] & & Y \arrow[dl] \arrow[dr] & \\ X & & Z & & X \end{tikzcd}$$

In the paper I express similar self links with the following compact notation

$$X \rightleftharpoons Z_4 \subset \mathbb{P}^4$$

In particular the Weil divisors group on Z is generated by Q and $\overline{Q}$. The two quadrics are conjugated under complex conjugation, so that over $\mathbb{R}$ they are not defined individually. In particular $Z/\mathbb{R}$ is $\mathbb{Q}$-factorial, hence birationally rigid by Theorem 5. Observe that X is not defined over $\mathbb{R}$.

Before explaining the proof of Theorem 2, let me just give a brief look at the determinantal quartic from the point of view of Sarkisov program. Let $X = (\det M = 0) \subset \mathbb{P}^4$, with M general. Consider a Laplace expansion of $\det M$ with respect to the j-th row. Then the equation of X has the form $\sum_i l_i A_{ji} = 0$, where the l_i are linear forms and the A_{ji} are cubic forms. Then the A_{ji}s generate the ideal of a smooth surface B^r_j of degree 6, a Bordiga surface. It is easy to see that B^r_j passes through all singular points of X. The latter are the rank two points therefore any order three minor has to vanish. Therefore X is not factorial and consequently not $\mathbb{Q}$-factorial (terminal Gorenstein $\mathbb{Q}$-factorial singularities are factorial). The symmetry between rows and columns, in the Laplace expansion, suggest that X is a midpoint of a Sarkisov link. Indeed this is the case of a well known "determinantal" involution of $\mathbb{P}^3$, [Pe],

$$\mathbb{P}^3 \rightleftharpoons X \subset \mathbb{P}^4$$

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1. Maximal singularities and the main theorem

I start rephrasing Theorem 2 in the following form.

Theorem 7. *Let $X_4 \subset \mathbb{P}^4_{\mathbb{C}}$ be a $\mathbb{Q}$-factorial quartic 3-fold with only ordinary double points as singularities. Then any birational map $\chi : X \dashrightarrow V$ to a Mori space V/T is a self map, i.e. $V \cong X$.*

To prove Theorem 7 I use the Maximal singularities method combined with Sarkisov Program, as described in [Co2], [CPR] and [CM$^{\mathrm{u}}$]. I rely on those papers for the very basic definitions like Mori fiber spaces, weighted projective spaces, Sarkisov program and links, and philosophical background. Here I quickly recall what is needed.

Definition 8 (degree of χ)**.** Suppose that X is a Fano 3-fold with the property that $A = -K_X$ generates the Weil divisor class group: $\mathrm{WCl}\, X = \mathbb{Z} \cdot A$ (this holds in our case under the $\mathbb{Q}$-factoriality assumption). Let $\chi\colon X \dashrightarrow V$ be a birational map to a given Mori fiber space $V \to T$, and fix a very ample linear system $\mathcal{H}_V$ on V; write $\mathcal{H} = \mathcal{H}_X$ for the birational transform $\chi_*^{-1}(\mathcal{H}_V)$.

The *degree* of χ, relative to the given V and $\mathcal{H}_V$, is the natural number $n = \deg \chi$ defined by $\mathcal{H} = nA$, or equivalently $K_X + (1/n)\mathcal{H} = 0$.

Definition 9 (untwisting)**.** Let $\chi\colon X \dashrightarrow V$ be a birational map as above, and $f\colon X \dashrightarrow X'$ a Sarkisov link. We say that f *untwists* χ if $\chi' = \chi \circ f^{-1}\colon X' \dashrightarrow V$ has degree smaller than χ.

Definition 10 (maximal singularity)**.** Let X be a variety and $\mathcal{H}$ a movable linear system. Suppose that $K_X + (1/n)\mathcal{H} = 0$ and $K_X + (1/n)\mathcal{H}$ has not canonical singularities. A *maximal singularity* is a terminal (extremal) extraction $f : Y \to X$ in the Mori category, see [CM$^{\mathrm{u}}$, §3], having exceptional *irreducible* divisor E such that $f^*(K_X + c\mathcal{H}) = K_Y + c\mathcal{H}_Y$, where $c < 1/n$ is the canonical threshold. The image of E in X, or the center $C(X, v_E)$ of the valuation v_E, is called the *center* of the maximal singularity.

Remark 11. *In this paper all maximal singularities will be either the blow up of an ordinary double point, or generically the blow up of the ideal of a curve $\Gamma \subset X$. In both cases this is the unique possible maximal singularity with these centers. This is easy for curves, while for an ordinary double point it is due to Corti,* [Co2, Theorem 3.10].

Lemma 12 ([CPR, Lemma 4.2])**.** *Let X, V/T, $\mathcal{H}$ be as before, $\chi : X \dashrightarrow V$ a birational map. If $E \subset Z \to X$ is a maximal singularity, any link $X \dashrightarrow X'$, starting with the extraction $Z \to X$, untwists χ.*

The above Lemma, together with Sarkisov program, allow to restrict the attention on maximal singularities. To study maximal singularities there is an invariant which is very often useful: the self intersection of the exceptional divisor. The next Lemma allow to compute E^3 when the center is smooth curve on X. To do this I have to determine the correction terms that are needed to make adjunction formula work in the presence of cA_1 singularities along Γ. This is done using the theory of Different developed in [U2, §16] and the following Lemma kindly suggested by Nikos Tziolas. In the statement and proof of the Lemma I need a notion of singularity for pairs curve and surface with A_t points.

Definition 13. Assume that $p \in S \sim (0 \in (xy + z^{t+1} = 0))$, for some $t \geq 1$, and $\Gamma \subset S$ is a smooth curve through p. Let $\nu : U \to S$ be a minimal resolution with exceptional divisors E_i, with $i = 1, \ldots, t$. Here I mean that the rational chain starts with E_1, ends with E_t, and for $1 < i < t$ the intersection $E_i \cdot E_j$ is non zero if and only $j = i \pm 1$.

I say that (Γ, S) is an A_t^k singularity if $C_U \cdot E_k = 1$ (here and all through the paper I decorate with $_T$ the strict transform of objects on a variety T). Observe that since C is smooth then $C_U \cdot E_i = 0$ for any $i \neq k$.

Lemma 14 ([Tz3]). *Let $(0 \in X)$ be a cA_1 singularity and $0 \in \Gamma \subset X$ a smooth curve through it. Let $f : Y \to X$ be a terminal extraction with center a smooth curve Γ and exceptional divisor E. Then f can be obtained from the diagram*

$$\begin{tikzcd} & Z \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ W \arrow[dr, "\nu"'] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

such that

i) *W is the blow up of X along Γ. The ν-exceptional divisors are a ruled surface E over Γ and $F \cong \mathbb{P}^2$ aver the singular point. Z is a $\mathbb{Q}$-factorialization of E and ψ contracts $F_Z \cong F$ to a point.*
ii) *$S_Y = f_*^{-1} S \cong S$, where S is a general section of X through Γ.*
iii) *(Γ, S) is an A_{2k-1}^k singularity.*

Proof. First prove that W is cA, [Ko]. Let S be the general section of X through Γ. Then one can assume, [Tz1], that S is given by $xy - z^{n+m} = 0$ and Γ by $x - z^n = y - z^m = 0$, for some $n \leq m$, equivalently S by $xy + xz^n + yz^m = 0$ and Γ by $x = y = 0$. Then X has the form

$$xy + xz^n + yz^m + tg_1(x, y, z, t) + tg_{\geq 2}(x, y, z, t) = 0$$

and Γ is $x = y = t = 0$. To have a cA_1 singularity the quadratic term $xy + tg_1(x, y, z, t)$ must be irreducible. Now a straightforward explicit computation of the blow up of the maximal ideal of Γ shows that W is cA.

Then by [Tz2] it follows that Z and hence Y, can be constructed in families. Therefore we may study the deformed equation

$$xy + xz^n + yz^m + tg_1(x, y, z, t) + t^k = 0$$

for $k \gg 1$. The blow up computation and the irreducibility of the quadratic term yields that W has isolated singularities along $E \cap F$. Therefore Z is just the blow up of E and hence $F_Z \cong F \cong \mathbb{P}^2$. This also proves that F_Z is contracted to a point by ψ.

To see the claim on S_Y take a general member $S_W \in |-K_W|$. Then S_W has A_i singularities and avoids the singular points along $E \cap F$. Let $C = S_W \cap F$. The C is contracted by ν and therefore $S_Y = \nu(S_W) \cong S$.

Since W is smooth on the generic point of $E \cap F$, [Tz1, Proposition 4.6], it follows that (Γ, S) is an A_{2k-1}^k singularity because in any other case W would be singular at $E \cap F$. $\square$

Next I derive the numerical result about self intersection from Lemma 14.

Lemma 15. *Let $f:Y\to X$ be a terminal extraction with center a smooth curve Γ and exceptional divisor E. Assume that X has only cA_1 points along Γ. Let Σ be any linear system with $\mathrm{Bsl}\,\Sigma=\mathcal{I}_\Gamma$ and $S\in\Sigma$ a general element. Assume that S is normal. Then $f_{|S_Y}:S_Y\to S$ is an isomorphism, and*

$$E^3=-S\cdot\Gamma-(\Gamma\cdot\Gamma)_S$$

or equivalently

$$E^3=K_X\cdot\Gamma-2g(\Gamma)+2-\mathrm{Diff}(\Gamma,S)$$

Remark 16. *In the hypothesis of Lemma 15 one can define the different of Γ in X as*

$$\mathrm{Diff}(\Gamma,X):=K_X\cdot\Gamma-E^3-2g(\Gamma)+2$$

This suggests the possibility to extend the theory of Different, [U2, §16]*, to higher codimension subvarieties.*

Proof. We already proved, Lemma 14, that $S_Y\cong S$. By hypothesis $f^*(S)=S_Y+E$, and consequently

$$E^3=f^*(S)\cdot E^2-S_Y\cdot E^2$$

Projection formula yields $f^*S\cdot E^2=-S\cdot\Gamma$. By Lemma 14 $E_{|S_Y}=\Gamma$, therefore $S_Y\cdot E^2=(E_{|S_Y}\cdot E_{|S_Y})_{S_Y}=(\Gamma\cdot\Gamma)_S$. Note that $K_S=(K_X+S)_{|S}$, therefore

$$(\Gamma\cdot\Gamma)_S=2g(\Gamma)-2-K_X\cdot\Gamma-S\cdot\Gamma+\mathrm{Diff}(\Gamma,S)$$

by adjunction formula compensated by the Different, [U2, Chapter 16]. □

I now go back to Theorem 7. The first task is to recognize birational maps. The geometry of X suggests the existence of some birational self maps, the "Italian" approach, according to [CPR]:

- the reflection through a singular point p
- the elliptic involution associated to a line l containing some singular point.

The general line through p intersect the quartic in two more points Q_1 and Q_2. The self map suggested is $Q_1\mapsto Q_2$. A general plane containing l has a smooth cubic C as residual intersection with X. Furthermore a singularity, say P, provides the family of these cubics of a section, namely a common origin to the group structure. The self map suggested is $R\mapsto -R$ where $-R$ is the inverse of R in the group structure on C with origin P.

Then I describe those maps in terms of Sarkisov links.

After [Co2], [CPR] and [CM$^{\mathrm{u}}$] this is now a nice and pleasant exercise. Indeed the only possibility that is not yet described in neither [Co2] nor [CM$^{\mathrm{u}}$] is the one of a line with three singularities along it. Assume that $l\subset X$ is a line with three distinct singular points along it. Note that this is the maximum number of singular points along a line on a quartic with isolated singularities. After a coordinate change we can assume that $l=(x_2=x_3=x_4=0)$ and the equation of X has the following form

$$F=L(x_0^2x_1+x_1^2x_0)+Q_1x_0^2+Q_2x_1^2+Q_3x_0x_1+C_1x_0+C_2x_1+D$$

Let $f : Y \to X$ be the unique terminal extraction with center l and exceptional divisor E. I want to understand the anticanonical ring of Y. Let

$$H_i = (x_i = 0)_{|X}$$

It is immediate that $H_{iY} \in |-K_Y|$. Since $l = (x_2 = x_3 = x_4 = 0)$ and f is generically the blow up of the maximal ideal then $-K_Y$ is nef. The linear form $\tilde{H} := L_{|X}$ has multiplicity two along l. Therefore a general plane section of $\tilde{H}$ through l has residual intersection a conic, say C, that, generically, intersects l in two points. In particular $C_Y \cdot K_Y = 0$ and $NE(Y) = \langle e, C\rangle$, where $e \subset E$ is f-exceptional. Note that the special hyperplane section $\tilde{H}_Y$ is covered by curves proportional to C. Therefore the ray spanned by $[C]$ is not small and by the two ray game I conclude that there is no Sarkisov link starting from the extraction $f : Y \to X$. This is usually called a bad link, [CPR], [Co2].

The only Sarkisov links that have center either a singular point or a line through a singular point are therefore the following:

ρ_x for any singular point $x \in X$

$$X \rightleftharpoons Z_6 \subset \mathbb{P}(1,1,1,1,3)$$

φ_1^l for any line $l \subset X$ passing through one singular point

$$X \rightleftharpoons Z_{12} \subset \mathbb{P}(1,1,1,4,6)$$

φ_2^l for any line $l \subset X$ passing through two singular points

$$X \rightleftharpoons Z_8 \subset \mathbb{P}(1,1,1,2,4)$$

Note that to a line with more than one singularity are associated different elliptic involutions. I can choose any singular point as origin on the elliptic curves. But still, by Sarkisov theory, the maximal singularity with center the line is unique. This is because the elliptic involution, in this case, is a composition elementary links.

To prove Theorem 7 it is now enough to show that any birational map can be factored by the self maps described. It is now standard, see [CPR, §3], that this is equivalent to the following.

Theorem 17. *Let $X_4 \subset \mathbb{P}^4_{\mathbb{C}}$ be a $\mathbb{Q}$-factorial quartic 3-fold with only ordinary double points and E a maximal singularity. Then either:*

- *the center $C(X, v_E) = p$ is a singular point, or*
- *the center $C(X, v_E) = l$ is a line through some singular point.*

In both cases the assignment identifies the maximal singularity, hence the Sarkisov link, uniquely, see Remark 11.

The proof of Theorem 17 is the core of the next section.

2. Exclusion

A maximal center on a Fano 3-fold is either a point or a curve.
The case of smooth points can be treated with many different techniques. The main result of [IM] is indeed that a smooth point is not a maximal center on a quartic. Corti, [Co2], gave an amazingly simple proof using numerical properties of linear system on surfaces. The recent classification of Kawakita, [Kw1], gives a third possible proof based on terminal extractions, see [Co1, Conjecture 4.7].

I am therefore bound to study centers of positive dimension. I can actually prove a stronger statement.

Theorem 18. *Let $X_4 \subset \mathbb{P}^4_{\mathbb{C}}$ be a $\mathbb{Q}$-factorial quartic 3-fold with only cA_1 points. Assume that a curve Γ is a center of maximal singularities. Then Γ is a line through some singular point.*

Remark 19. *The Theorem is an important step in the direction of* [CM, Conjecture 1.3]

Proof. From now on $\Gamma \subset X$ will be an irreducible curve assumed to be the center of a maximal singularity. The unique terminal extraction is then generically the blow up of the ideal of Γ in X. Therefore the linear system $\mathcal{H} \subset |\mathcal{O}(n)|$, associated to the extraction, satisfies

$$\gamma = \operatorname{mult}_\Gamma \frac{1}{n}\mathcal{H} > 1.$$

We prove the theorem in several steps

Step 1: A raw argument shows that $\deg \Gamma \leq 3$.
Step 2: Γ can not be a space curve.
Step 3: If Γ is a plane curve then it is a line through some singular point.

STEP 1. Choosing general members H_1, H_2 of $\mathcal{H}$ and intersecting with a general hyperplane section S we obtain

$$4n^2 = H_1 \cdot H_2 \cdot S > \gamma n^2 \deg \Gamma.$$

This implies that $\deg \Gamma \leq 3$.
STEP 2: SPACE CURVES. If Γ is a space curve, then by Step 1 it must be a rational normal curve of degree 3, contained in a hyperplane $\Pi \cong \mathbb{P}^3 \subset \mathbb{P}^4$. Let $S \in |I_{\Gamma,X}(2)|$ be a general quadric vanishing on Γ,$\mathcal{L}$ the mobile part of $\mathcal{H}_{|S}$; write

$$\mathcal{O}_S(1) = \frac{1}{n}\mathcal{H}_{|S} = L + \gamma\Gamma,$$

where $L = (1/n)\mathcal{L}$ is nef. Note that, because I_Γ is cut out by quadrics,

$$\operatorname{mult}_\Gamma \mathcal{H} = \operatorname{mult}_\Gamma \mathcal{H}_{|S} = n\gamma > n.$$

Let $f : Y \to X$ be the maximal singularity, with exceptional divisor E, and center Γ. By Lemma 15 we can compute E^3 by means of $\operatorname{Diff}(\Gamma, S)$. Assume that (C, U) is an A^k_t singularity, keep in mind Definition 13, let $\nu^*(C) = C_W + \sum d_i E_i$ then it is a straightforward check on the intersection matrix of an A_t singularity, see for instance [Ja, pg 16], that

$$(t+1)d_i = \begin{cases} i(t-k+1) & \text{if } i \leq k \\ (t-i+1)k & \text{if } i \geq k \end{cases} \tag{1}$$

Incidentally observe that $\operatorname{Diff}(C, U)_x = C_W \cdot \sum d_i E_i = d_k$.

I now come back to our original situation by Lemma 14 part iii) (Γ, S) is an A^k_{2k-1} singularity for some k, with $l/2 \geq k \geq 1$. In particular the Different is

$$\operatorname{Diff}(\Gamma, S)_p = k/2 \tag{2}$$

This proves, together with Lemma 15, that

$$E^3 = -3 + 2 - \sum_{p_i \in \operatorname{Sing}(S)} k_i/2 \tag{3}$$

Then we need to bound the contributions of the singularities globally.

Lemma 20. *In the above notation* $\sum_{p_i\in \mathrm{Sing}(S)} k_i\leq 7$.

Proof. To prove the bound we need the following reinterpretation of the k_i's, see also [Tz1]. By Lemma 14 part iii) we can realize $p_i\in\Gamma\subset S\subset\mathbb{Q}^3$ analytically as

$$0\in(x=y=0)\subset(xy+yz^{k_i}+xz^{k_i}=0)\subset\mathbb{C}^3,$$

see for instance [Ja, pg 13]. Let $\mu_i:Z\to\mathbb{C}^3$ be the blow up of $(x=y=0)$, with exceptional divisor E_Z and $F_i=\mu_i^{-1}(0)$. Then $S_{Z|E_Z}=k_iF_i+\text{effective}$. Let $\nu:W\to\mathbb{P}^4$ be the blow up of Γ, with exceptional divisor E_W, and $F_i=\nu^{-1}(p_i)$. Since $\mathrm{Bsl}\,|\mathcal{I}_{\Gamma,\mathbb{P}^4}(2)|=\Gamma$ then $X_{W|E_W}=k_iF_i+\text{effective}$. For any divisor $D\subset\mathbb{P}^4$ such that $D\supset\Gamma$ and $D_{|X}$ is smooth on the generic point of Γ we have

$$(D_Z\cap X_Z)_{|E_Z}=h_iF_{i|D_Z}+\text{effective},$$

for some

$$h_i\geq k_i \tag{4}$$

Thus to bound the global contribution it is enough to understand the normal bundle of Γ in some smooth divisor D such that $D_{|X}$ is smooth on the generic point of Γ. Let $H=\Pi_{|X}$ be the unique hyperplane section containing Γ, and $H_{|S}=\Gamma+\Delta$. I claim that $\Delta\not\supset\Gamma$. Assume the opposite and let $H_{|S}=2\Gamma+R$. Then $\deg R=2$ and R is a pair of skew lines, say, l_1, l_2, secant to Γ. Since S is general then $l_i\not\subset\mathrm{Bsl}\,\mathcal{H}$ and $l_i\cap\Gamma\not\subset(\mathrm{Sing}(X)\cap\Gamma)$. Then we derive the impossible

$$1=\mathcal{H}/n\cdot l_i\geq\gamma\Gamma\cdot l_i>\gamma.$$

We can therefore choose $D=\Pi$. It is well known, [Hu], that $N_{\Gamma/\mathbb{P}^3}\cong\mathcal{O}(5)\oplus\mathcal{O}(5)$. Let $\nu:W\to D$ be the blow up of Γ with exceptional divisor $E_W\cong\mathbb{F}_0$. Then $X_{W|E_W}\equiv f_0+7f_1$, where f_1 is a fiber of ν. The inequality $\sum k_i\leq\sum h_i\leq 7$ is obtained. $\square$

Consider again, the hyperplane section $H=\Pi_{|X}$ and the maximal singularity $f:Y\to X$. Let $D_Y\subset Y$ be any effective irreducible divisor distinct from E. Then $D_Y=f^*D-\alpha E$, for some positive $\alpha\in\mathbb{Q}$ and $D\in|\mathcal{O}(d)|$. The divisor D_Y is numerically equivalent to $dH_Y+(d-\alpha)E$. To conclude the step it is, therefore enough to prove that the cone of effective divisors on Y is generated by H_Y and E. This is the content of the next Lemma.

Lemma 21. $NE^1(Y)=\langle H_Y,E\rangle$.

Remark 22. *This is just a rewriting of the usual exclusion trick. I prove that a linear system like $\mathcal{H}$ has to have a fixed component, in this case H. I hope that in this way it is easier to digest and maybe generalize. See also Remark 24.*

Proof. Let $B_Y\subset Y$ be any effective irreducible $\mathbb{Q}$-divisor distinct from E and H_Y. Then $B_Y=f^*B-\beta E$, for some positive $\beta\in\mathbb{Q}$ and $B\in|\mathcal{O}(b)|$. Actually $\beta\in\mathbb{Z}$ since X has index 1 and is $\mathbb{Q}$-factorial. I have to prove that $\beta\leq b$. By Lemma 15 $\dim\mathrm{Bsl}\,|S_Y|\leq 0$, hence the cycle $B_Y\cdot H_Y\cdot S_Y$ is effective. The following inequality is satisfied

$$\begin{aligned}0\leq\ & B_Y\cdot H_Y\cdot S_Y=(f^*(B)-\beta E)(f^*(H)-E)(f^*S-E)\\ =\ & 8b-3b-9\beta-\beta E^3=5b-(8-\textstyle\sum k_i/2)\beta\end{aligned}$$

This proves the claim for $\sum k_i\leq 6$.

Assume that $\sum k_i=7$. First I need to better understand this special configuration of singularities.

Let $\nu : W \to \Pi$ be the blow up of Γ with exceptional divisor $E_W \cong \mathbb{F}_0$, g an "horizontal" ruling of E_W, and f_i fibers of ν. Then the assumption on singularities yield $H_{W|E_W} = g + \sum_1^7 f_i$, where the f_i are not necessarily distinct. Note that for each point $y \in \Gamma$ there is a quadric cone $Q^y \subset \Pi$ containing Γ and with vertex y. Then $Q^y_{W|E_W} = g_y + f$. In particular for any $g \subset E_W$ "horizontal" ruling there exists a quadric cone $Q^g \subset \Pi$ such that $Q^g_W \supset g$. This proves that there exists a quadric cone, say $\tilde{Q} \subset \mathbb{P}^3$, such that $\tilde{Q}_{|H} = 2\Gamma + C$, for some conic C. Similarly there exists a cubic surface $\tilde{M}$ such that $\tilde{M}_{|H} = 2\Gamma + R$ and $\tilde{M}_{|\tilde{Q}} = 2\Gamma$. Therefore the equation of H can be written as

$$\tilde{Q}K + \tilde{M}P = 0,$$

where K is a quadric and P is a linear form.

Assume that $\Pi = (x_4 = 0)$. Let Σ be the linear system of quadrics spanned by $\{\tilde{Q}, x_4x_0, \ldots, x_4x_3\}$. Fix $\overline{S} \in \Sigma_{|X}$ a general element. By construction we have $H_{|\overline{S}} = 2\Gamma + C$. Let

$$H_{Y|E} = \Gamma_0 + F,$$

where F is f-exceptional. Then for effective, f-exceptional divisors F' and G, we have

$$\overline{S}_{Y|E} = \Gamma_0 + F', \quad \overline{S}_{Y|H_Y} = \Gamma_0 + C + G$$

and

$$(\overline{S}_Y - E) \cdot H_Y = C + G - F \tag{5}$$

Claim 1. $F - G \equiv \mathcal{O}_E$

Proof of the Claim. The cycle F is the f-exceptional part of H_Y. The cycle G is f-exceptional and it is contained in $\operatorname{Bsl}\Sigma_Y$ therefore $F - G$ is effective. Let $\phi := f_{|E} : E \to \Gamma$ be the restriction morphism and $E^0 = E \setminus \phi^{-1}(\operatorname{Sing}(X) \cap \Gamma)$. In our notation we have $\operatorname{Bsl}(\Sigma_{Y|E}) = \Gamma_0 + G$, thus we can assume that $F' = G + M$, for some divisor $M = \phi^*A$ supported on E^0. Let me interpret this divisor in a different way. Let $\overline{Q} \in \Sigma$ be the quadric whose to X is $\overline{S}$. Since $\tilde{Q} = \overline{Q}_{|\Pi}$ is a cone then $N_{\Gamma/\overline{Q}} \cong \mathcal{O}(2) \oplus \mathcal{O}(5)$. Let $\nu : W \to \overline{Q}$ be the blow up of Γ, with exceptional divisor E_W. Then a computation similar to that of Lemma 20 yields

$$X_{W|E_W} \equiv \Gamma_0 + \nu^*_{|E_W}\mathcal{O}(10) \text{ and } X_{W|E_W} = \Gamma_0 + \sum h_i f_i + \text{effective}$$

where the $h_i \geq k_i$ and $\nu(f_i) \in \operatorname{Sing}(X)$. This proves that $\deg A \leq 10 - \sum k_i = 3$. Taking into account a reducible quadric in Σ, we have $F' \equiv F + \phi^*\mathcal{O}(3)$. This shows that $F - G \equiv \phi^*(A - \mathcal{O}(3))$ and together with the bound on the degree of A the desired $F - G \equiv \mathcal{O}_E$. $\square$

Projection formula and equation (3) at page 8 yield

$$(\overline{S}_Y - E) \cdot H_Y \cdot E = 6 - 2H_Y \cdot E^2 = 6 - 2(f^*H \cdot E^2 - E^3) = 3$$

Then by Claim 1 and equation (5) I derive

$$E \cdot C = 3 \tag{6}$$

Note that if C is reducible, then each irreducible component C_i is a line. In this case the inequality $E \cdot C_i \leq 2$ is immediate. Thus we proved that for any irreducible component $C_i \subset C$

$$E \cdot C_i \geq \deg C_i \tag{7}$$

Let us assume that C is irreducible, the reducible case is similar and left to the reader.

Assume that $B_{Y|H_Y} = aC + \Delta$, for some effective divisor Δ, with $\Delta \not\supset C$. The above construction gives

$$(f^*(B - a\overline{S}) - (\beta - 2a)E)_{|H_Y} = \Delta + a(F - G)$$

This proves that $(f^*(B - a\overline{S}) - (\beta - 2a)E) \cdot C \geq 0$ and we conclude by equation (7) that

$$(b - 2a) \geq (\beta - 2a)$$

$\square$

Step 3: plane curves. Here we assume that Γ is a plane curve of degree d (by Step 1, $d \leq 3$), other than a line passing through some singular point. Let $\Pi \subset \mathbb{P}^4$ be the plane spanned by Γ. Fix S, S' be general members of the linear system $|\mathcal{I}_{\Gamma,X}(1)|$. Here it is helpful and convenient to treat two cases, namely:

Case 3.1: $\Gamma \cap \operatorname{Sing}(X) = \emptyset$, and $1 \leq d \leq 3$.

Case 3.2: $\Gamma \cap \operatorname{Sing}(X) \neq \emptyset$ and $2 \leq d \leq 3$.

Case 3.1. I first deal with the easy, and well known, case of curves in the smooth locus. Let $f : Y \to X$ be the maximal singularity with center Γ, and exceptional divisor E. Then Y is just the blow up of $\mathcal{I}_\Gamma$ and by Cutkosky's classification, [Cu], of terminal extraction

$$E^3 = K_X \cdot \Gamma - 2p_a(\Gamma) + 2$$

Lemma 23. $NE^1(Y) = \langle S_Y, E\rangle$

Proof. Let $B_Y \subset Y$ be any effective irreducible $\mathbb{Q}$-divisor distinct from E and S_Y. Then $B_Y = f^*B - \beta E$, for some positive $\beta \in \mathbb{Z}$ and $B \in |\mathcal{O}(b)|$. The claim is equivalent to prove that $\beta \leq b$. Consider a general element $D \in |I_{\Gamma,X}(d)|$. The cycle $B_Y \cdot S_Y \cdot D_Y$ is effective, thus

$$\begin{aligned} 0 \leq \quad & B_Y \cdot S_Y \cdot D_Y = (f^*B - \beta E)(f^*S - E)(f^*B - E) \\ = \quad & 4bd - bd - d\beta - d^2\beta - \beta E^3 = 3bd - d^2\beta + (2p_a(\Gamma) - 2)\beta \end{aligned}$$

It is a simple check that for any possible pair $(d, p_a(\Gamma))$ the equation gives $\beta \leq b$. $\square$

The Lemma finish off the Case 3.1.

Case 3.2. From now on we assume that there are singular points along Γ and Γ is not a line.

We work with the linear system $\Sigma = |S, S'|$, even though Γ is usually only a component of its base locus $C = S \cap S' = \operatorname{Bsl}\Sigma = X \cap \Pi$. Write

$$C = \mu\Gamma + \sum \mu_i \Gamma_i$$

We are assuming that X is $\mathbb{Q}$-factorial. This implies that Π can not be contained in X, and C is a *curve*. Assume first that the intersection $S \cdot S'$ is reduced then $\operatorname{mult}_\Gamma \mathcal{H} = \operatorname{mult}_\Gamma \mathcal{H}_{|S}$ and $\operatorname{mult}_{\Gamma_i} \mathcal{H} = \operatorname{mult}_{\Gamma_i} \mathcal{H}_{|S}$. We always restrict to S and write

$$\begin{aligned} A := (1/n)\mathcal{H}_{|S} \quad &= L + \gamma\Gamma + \textstyle\sum \gamma_i \Gamma_i \\ S'_{|S} \quad &= C = \Gamma + \textstyle\sum \Gamma_i \end{aligned}$$

The technique consists in selecting a "most favorable" component of C, performing an intersection theory calculation using that L is nef, and get that $\gamma \leq 1$. This

inequality contradicts the hypothesis that Γ is a maximal singularities. Indeed, keeping in mind Remark 11, we have

$$n < \operatorname{mult}_\Gamma \mathcal{H} = \operatorname{mult}_\Gamma \mathcal{H}_{|S} = \gamma n \leq n$$

Remark 24. *This is similar to what I did with the twisted cubic with $\sum k_i = 7$, see Lemma 21. I believe that the E^3 approach works also in this case, but I did not check it. On the other hand each different configuration needs different calculations. For this reason I developed a unified approach with more emphasis on the intersection theory on S.*

Because Γ is a center of a *maximal* singularity, $\gamma \geq \gamma_1$, γ_2, hence possibly after relabeling components of C, we can assume that:

$$\gamma \geq \gamma_2 \geq \gamma_1.$$

Consider now the effective $\mathbb{Q}$-divisor

$$(A - \gamma_1 S')_{|S} = L + (\gamma - \gamma_1)\Gamma + (\gamma_2 - \gamma_1)\Gamma_2.$$

I now show that $(\Gamma \cdot \Gamma_1)_S \geq \deg \Gamma_1$; together with the last displayed equation this implies that $\gamma \leq 1$ and finishes the proof.

The curve Γ_1 is either a line or a conic. Let $D \in |\mathcal{I}_{\Gamma_1,\mathbb{P}^4}(\deg \Gamma_1)|$ a general element, and $D_{|S} = \Gamma_1 + F$. Since $D \cap \Pi = \Gamma_1$ is a complete intersection, then F intersects Γ only at $\operatorname{Sing}(X) \cap \Gamma_1 \cap \Gamma$. Fix a point $p \in \operatorname{Sing}(X) \cap \Gamma \cap \Gamma_1$. Assume that $p \in X \sim (0 \in (xy + z^2 + t^l = 0))$.

Let $f_1 : X_1 \to X$ be the blow up of $p \in X$ with exceptional divisor E_1. Then $S_{1|E_1} = C_1$ is a conic and since $\operatorname{Bsl} \Sigma = \Pi$ then C_1 is reduced. This proves that either $S_{1|E_1}$ is smooth or it has one singular point only, say x_1, and C is a pair of lines. Let $f_2 : X_2 \to X_1$ be the blow up of x_1, with exceptional divisor E_2. If $p_1 \in X_1$ is a smooth point then E_2 is a plane, and $\operatorname{Bsl} \Sigma_2$ is contained in a line. The surface S_2 is smooth and already S_1 was non singular. Otherwise $p_1 \in X_1 \sim (0 \in (xy + z^2 + t^{l-2} = 0))$ and we simply repeat the same argument. This gives a morphism $\nu : W \to X$, with exceptional divisors G_i, for $i = 1, \ldots, g$. Such that $\nu_{|S_W} : S_W \to S$ is a minimal resolution. Moreover $S_W \cap G_i = L_i \cup R_i$ is a pair of disjoint (-2)-curves, for any $i < g$, and $S_W \cap G_g = T$ is either a (-2)-curve or a pair of (-2)curves intersecting in a point. This proves that $p \in S_1$ is an A_m point, with $m \leq l$. Furthermore F is smooth at x.

Number all irreducible components of the resolution $\nu_{|S_W}$ from 1 to $m = 2g - \epsilon$, where $\epsilon = 1, 0$, according to the parity of m. Start with $L_1 =: E_1$, then $L_i =: E_i$ and $R_i = E_{m+1-i}$ for any $i < g$. Similarly let $T = L_g \cup R_g =: E_g \cup E_{g+1}$, where $E_g \cdot E_{g+1} = 1$, if it is reducible and $T = E_g$ if it is irreducible.

As our aim is to calculate an intersection product we need to understand the pairs (Γ, S), (Γ_1, S), and (F, S).

If $(\Gamma_1)_W \cap T = \emptyset$ then there exists an index $j < g$ such that $(\Gamma_1)_W \cdot E_j = 1$ and $F_W \cdot E_{m+1-j} = 1$. If $(\Gamma_1)_W \cap T \neq \emptyset$ and $T = L_g \cup R_g$ is reducible we labeled the component in such a way that $(\Gamma_1)_W \cdot E_g = 1$ and $F_W \cdot E_{g+1} = 1$. Finally for $(\Gamma_1)_W \cap T \neq \emptyset$ and T is irreducible then $(\Gamma_1)_W \cdot E_g = F_W \cdot E_g = 1$.

In any case (Γ_1, S) is of type A_m^j, for some $j \leq m + 1 - j$. While (F_Z, S) is of type A_m^{m+1-j}.

Let

$$\nu_{|S_W}^*(\Gamma_1) = (\Gamma_1)_W + \sum r_i E_i$$

and

$$\nu^*_{|S_W}(F) = F_W + \sum f_i E_i.$$

Then the r_is and the f_is are completely determined by equation (1) at page 8. The index j satisfies the inequality $j \leq m+1-j$ by hypothesis. Assume that $i \leq m+1-i$ is also true, then $m+1-j \geq i$. Thus for any index i such that $i \leq m+1-i$ we have,

$$(m+1)(r_i - f_i) = \begin{cases} (m+1-2j)i & \text{if } i \leq j \\ (m+1-2i)j & \text{if } i \geq j \end{cases}$$

These yield

$$r_i \geq f_i \text{ for any } i \leq m+1-i. \tag{8}$$

The curve $\Gamma \subset \Pi$ has at most a simple node or a simple cusp then

$$\Gamma \cdot E_i = 0 \text{ for any } i > g \text{ i.e. } i > m+1-i \tag{9}$$

By construction $F_W \cdot \Gamma_W = 0$ therefore by projection formula

$$(\Gamma_1 \cdot \Gamma - F \cdot \Gamma)_x \geq \sum_i (r_i - f_i)\Gamma_W \cdot E_i.$$

By equation (9) we can restrict the summation on indexes satisfying $i \leq m+1-i$ and equation (8) yields

$$(\Gamma_1 \cdot \Gamma)_x - (F \cdot \Gamma)_x \geq 0.$$

Finally all contributions coming from singular points give

$$\deg \Gamma_1 \deg \Gamma = D \cdot \Gamma = ((\Gamma_1 + F) \cdot \Gamma)_S \leq 2(\Gamma_1 \cdot \Gamma)_S,$$

and consequently the needed bound since $\deg \Gamma \geq 2$.

Next we consider the case in which $S'_{|S} = \Gamma + 2l$, where Γ is a conic and l is a line. Again S is smooth at $(\Gamma \cap \Gamma_1) \setminus (\mathrm{Sing}(X) \cap \Gamma)$ as well as on the generic point of l. Indeed we are just fixing a plane, therefore we can always choose an hyperplane containing Π, and not tangent to X at both $(\Gamma \cap \Gamma_1) \setminus (\mathrm{Sing}(X) \cap \Gamma)$ and at the generic point of l. Then $\mathcal{H}_{|S} = \mathcal{L} + \gamma\Gamma + \alpha l$. Consider the $\mathbb{Q}$-divisor

$$(\mathcal{H} - (\alpha/2)S')_{|S} = \mathcal{L} + (\gamma - \alpha/2)\Gamma.$$

Then

$$(1 - (\alpha/2)) \geq (\gamma - \alpha/2)\Gamma \cdot l.$$

To exclude this case we argue exactly as before that $\Gamma \cdot l \geq 1$. Keep in mind that also in this case S has only isolated singularities. Therefore locally around x all the calculations are the same.

Finally we have to treat the double conic case. That is assume that $S'_{|S} = 2\Gamma$. If there exists an hyperplane section $\tilde{S}$ such that $\mathrm{mult}_\Gamma \tilde{S} = 2$ then for a general hyperplane section H

$$4 = H \cdot \frac{\mathcal{H}}{n} \cdot \tilde{S} \geq \frac{4}{n} \mathrm{mult}_\Gamma \mathcal{H} = 4\gamma.$$

We can therefore assume that the tangent space to X along $\Gamma \setminus (\Gamma \cap \mathrm{Sing}(X))$ is not fixed. It is immediate to observe that for any smooth point $p \in \Gamma$ the embedded tangent space contains Π. Let us assume the following notations:

- $\Gamma \subset \Pi \subset \mathbb{P}^4 \sim (x_0x_4 + x_3^2 = 0) \subset (x_1 = x_2 = 0) \subset \mathbb{P}^4$,
- $x \equiv (1:0:0:0:0) \notin \mathrm{Sing}(X)$,
- $T_xX = (x_1 = 0)$,

- $S=(x_1=0)_{|X}$,
- $S'=(x_2=0)_{|X}$.

By construction $\mathcal{H}_{|S}=g\Gamma+\mathcal{L}$, where $\mathcal{L}$ is a linear system without fixed components and $g\geq\gamma$. Up to consider $2\mathcal{H}$ we can further assume that $g=2k$ is even. Since $S\cdot S'=2\Gamma$ then a general divisor $H\in\mathcal{H}$ has an equation of type

$$H=(x_2^kL_1+x_1^kL_2)_{|X},$$

where $\deg L_i\geq 1$. Let $y\equiv(0:0:0:0:1)$, we can assume without loss of generality that $y\notin\mathrm{Sing}(X)$, $T_yX=(x_1+x_2)$ and $L_1(y)\neq 0$, $L_2(y)\neq 0$. The equation of X is of the form

$$(x_0x_4+x_3^2)^2+x_1F_1+x_2F_2=0,$$

to express X at the point y, in a better way, we can rewrite it as follows

$$x_4^3(x_1+x_2)+x_4^2(x_0^2+x_1R_1+x_2R_2)+x_4C+D=0.$$

Let $F=(x_2^kL_1+x_1^kL_2=0)$ I claim that due to the monomial $x_0^2x_4^2$

$$\mathrm{mult}_y F_{|X}\leq k+1\leq\deg F.$$

Indeed let $\nu:Y\to\mathbb{P}^4$ be the blow up of the point y. Let y_i be the coordinates in the exceptional divisor E_0 of equation $(x_3=0)$ in the affine piece $y_3\neq 0$. Then

$$F_Y=(y_2^kL_1'+y_1^kL_2'=0),\ \text{and}\ \ X_Y=(y_1+y_2+(y_0^2+y_1R_1'+y_2R_2')x_3+G'x_3^2),$$

and

$$\mathrm{mult}_y F=k.$$

Let $\mu:W\to Y$ be the blow up of $G=X_{Y|E_0}$, with equations $x_3=y_1+y_2=0$. Let t be the coordinate in the exceptional divisor E_1 of equation $(x_3=0)$. The polynomial L_i does not vanish at y then $F_{Y|E_0}=(\alpha y_1^k+\beta y_2^k)$, for some non zero numbers α and β. Therefore $(y_1+y_2)^2$ does not divide $(\alpha y_1^k+\beta y_2^k)$, and

$$\mathrm{mult}_G F_Y\leq 1.$$

If $\mathrm{mult}_G F_Y=1$ then

$$F_W=(tA_0+y_1A_1+y_2A_2+x_3B)\ \ \text{and}\ \ X_W=(t+y_0^2+M+Nx_3),$$

for non zero polynomials A_i, and B, and due to the presence of the monomial y_0^2 the divisor $F_{W|E_1}$ does not contain $X_{W|E_1}$ and consequently $\mathrm{mult}_x F_{|X}\leq k+1$. This inequality concludes the proof. □

Remark 25. *I proved that a double conic is never the center of maximal singularities on any terminal $\mathbb{Q}$-factorial quartic. This relax the assumptions in* [CM, Theorem 1.1].

It is still left to adapt the proof to arbitrarily fields of characteristic 0.

Proof of Theorem 5. Let again Γ be a center of maximal singularities for the linear system $\mathcal{H}\subset|\mathcal{O}(n)|$. If Γ is defined over k then all the proof works exactly as in the algebraically closed field case. The only observations I want to add are the following. When Γ is a twisted cubic then $\Pi\cong\mathbb{P}^3\supset\Gamma$ is defined over k. Moreover $H=\Pi_{|X}$ has to be smooth on the generic point of Γ, as in the proof of Lemma 20, and hence irreducible, by $\mathbb{Q}$-factoriality. When Γ is a plane curve of degree greater than 1, the plane $\Pi\supset\Gamma$ is defined over k, and $\Pi\cap X$ is a curve.

Assume that Γ is not defined over k, and let $r=\deg[k(\Gamma):k]$. If $\Gamma=P$ is a point then $4n^2=\mathcal{H}^2\cdot\mathcal{O}(1)\geq r(\operatorname{mult}_P\mathcal{H})^2$. If P is smooth then $\operatorname{mult}_P\mathcal{H}^2>4n^2$, [Co2, Theorem 3.1], while for singular P, $\operatorname{mult}_P\mathcal{H}>n$, [Co2, Theorem 3.10], and consequently $\operatorname{mult}_P\mathcal{H}^2>2n^2$, the exceptional divisor is a quadric. This proves that when $r\geq 2$ no point can be a center of maximal singularities.

If Γ is a curve then again by numerical reasons

$$4n=\mathcal{H}\cdot\mathcal{O}(1)^2\geq r\deg\Gamma\operatorname{mult}_\Gamma\mathcal{H}>2n\deg\Gamma,$$

so that Γ is a line and $r\leq 3$. Let Γ_i the conjugate lines over k. First observe that $\Gamma\cap\Gamma_i\neq\emptyset$. Indeed they are both centers of maximal singularities on $\overline{k}$ and we can untwist Γ over $\overline{k}$. If Γ_i is disjoint from Γ the untwist is an isomorphism on the generic point of Γ_i. This is very clear from our description in terms of Sarkisov links. Then its strict transform is a curve, say Γ_i', of degree $g>1$. Let $\mathcal{H}'$ be the untwist of $\mathcal{H}$, then by Lemma 12, $\mathcal{H}'\in|\mathcal{O}(n')|$, for some $n'<n$. But then $\operatorname{mult}_{\Gamma_i'}\mathcal{H}'>n$ and this is not allowed by the proof of Theorem 18.

To conclude we have to study conjugate lines intersecting in a point. Assume that $r=2$, denote $\Pi\subset\mathbb{P}^4$ the plane spanned by Γ and Γ_1. Let S, S' be general members of the linear system $|\mathcal{I}_{\Pi,X}(1)|$. Observe that Π is defined over k, therefore $\Pi\cap X=(\Gamma+\Gamma_1)+\Delta$ is a curve. By the proof of Theorem 17, since X has only ordinary double points, all singular points of S are of type $0\in(xy+z^{t+1}=0)$, with $t\leq 2$.

Assume that $\Pi_{|X}\neq 2(\Gamma+\Gamma_1)$ then following the same arguments of page 11 we have to prove that for any irreducible curve $C\subset\Delta$

$$(\Gamma+\Gamma_1)\cdot C\geq\deg C$$

Fix a point $p\in C\cap\Gamma$. Since both C and Γ are curves contained in Π and $p\in S\sim(0\in(xy+z^t=0))$, with $t\leq 3$, then

$$(C\cdot\Gamma)_x\geq\frac{1}{2}.$$

Similarly for Γ_1, so that

$$C\cdot(\Gamma+\Gamma_1)\geq 2\frac{1}{2}\deg C\geq\deg C.$$

If $\Pi\cap X=2(\Gamma+\Gamma_1)$ then, up to a projectivity, we can write the equation of $X/\overline{k}$ as

$$x_0^2x_4^2+x_1F_1+x_2F_2=0.$$

Then we derive a contradiction as in the double conic case. Keep in mind that the crucial point was the presence of the monomial $x_0^2x_4^2$.

Note that the two lines are centers of maximal singularities on $\overline{k}$. Here we proved that they are not centers of maximal singularities with the same associated linear system. The case $r=3$ is similar. If all lines stays on the same k-plane I conclude as above. If they span a $\mathbb{P}^3$ say Π, then Π is defined over k. Moreover $H=\Pi_{|X}$ has to be smooth on the generic point of the lines, as in the proof of Lemma 20. Therefore the plane spanned by each pair of lines is not contained in X and I conclude as before. $\square$

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